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Non-invariant two-loop counterterms for the three-gauge-boson vertices

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Abstract: Some practical applications of algebraic renormalization are discussed. In particular we consider the two-loop QCD corrections to the three-gauge-boson vertices involving photons, Z and W bosons. For this purpose also the corresponding two-point functions have to be discussed. A recently developed procedure is used to analyze the breaking terms of the functional identities and explicit formulae for the universal counterterms are provided. Special attention is devoted to the treatment of infrared divergences.

Keywords: Renormalization Regularization and Renormalons, Gauge Symmetry, Standard Model, NLO Computations.

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Contents

1. Introduction

The impressive experimental precision mainly reached at the electron-positron colliders LEP and SLC and at the proton anti-proton collider Tevatron has made it mandatory to evaluate higher order quantum corrections. The dominant contributions arise from perturbative calculations in the Standard Model (SM) of elementary particle physics and some of its extensions. As the momentum integrals occurring within the usual evaluation of quantum corrections are divergent, a regularization accompanied by a renormalization prescription is adopted. Due to chiral couplings involving γ_5, no invariant regularization scheme is known for the Standard Model — leaving aside the lattice regularization with the Ginsparg-Wilson version of chiral symmetry [1]. The practicality of the latter scheme for higher-loop calculations has to be explored.

It is well known that in the framework of dimensional regularization only the non-invariant 't Hooft-Veltman scheme for γ_5 is shown to be consistent to all orders [2, 3]. The naive dimensional scheme (NDR) leads to inconsistencies in connection to γ_5 and the higher order calculations within the SM have already reached a point where these inconsistencies cannot be avoided. In [4] it was emphasized that the NDR scheme can still be used in many specific calculations and also a practical modification of the NDR scheme was proposed. In this paper we want to advertise an efficient consistent calculation using a non-invariant regularization scheme. This has the consequence that in general the functional identities like the Ward-Takahashi (WTI) and the Slavnov-Taylor identities (STI) are violated by local breaking terms. However, the concept of algebraic renormalization provides a powerful tool to fix the identities and remove the breaking terms (see, e.g. [5]).

In a recent paper, algebraic renormalization has been considered with regard to practical applications [6]. A procedure has been suggested and worked out, which allows an efficient determination of the breaking terms. Actually the computation can be reduced to the evaluation of universal, i.e. regularization-scheme-independent, counterterms.

In this letter we want to apply the method to the three-gauge-boson vertices involving two W bosons and a photon (AWW) or Z boson (ZWW), respectively. They constitute a building block to the important W pair production process in e^+e^- annihilation, which plays a crucial role at LEP2. Furthermore we consider the vertex functions involving three neutral gauge bosons, which we will denote by ZAA, AZZ and ZZZ. Note that AAA vanishes because of Fury's theorem. Also in the context of anomalous couplings the precise study of the three-gauge-boson vertices is of importance.

In [7] the one-loop diagrams contributing to $e^+e^- \to W^+W^-$ have been computed in the framework of dimensional regularization. However, proceeding to higher orders, a consistent treatment of γ_5 becomes mandatory and the popular, naive dimensional regularization has to be given up. The method of algebraic renormalization provides the possibility to adopt any convenient regularization — it only has to be consistent.

Our aim is to focus on two-loop QCD corrections, which has the consequence that at the one-loop order only the fermionic contributions have to be considered. Furthermore we decided to work in the framework of the background field gauge, which has the advantage that only WTIs with external background fields (and no STIs) have to be considered at the highest order. They have the same structure to any order in perturbation theory.

Let us in the following briefly review the main steps elaborated in [6] to remove the breaking terms. The use of a non-invariant regularization scheme induces breaking terms into the STIs

$$[\mathcal{S}\,(\Gamma)]^{(n)} = \hbar^n \Delta_S^{(n)} + \mathcal{O}(\hbar^{n+1})\,, \tag{1.1}$$

which implement the Becchi-Rouet-Stora-Tyutin (BRST) symmetry [8], and into the WTIs

$$\mathcal{W}_{(\lambda)}\left(\Gamma^{(n)}\right) = \hbar^n \Delta_W^{(n)}(\lambda) + \mathcal{O}(\hbar^{n+1})\,, \tag{1.2}$$

which implement the background gauge invariance of the SM. The local breaking terms are denoted by $\Delta_S^{(n)}$ and $\Delta_W^{(n)}(\lambda)$. Note that the locality is a consequence of the Quantum Action Principle (QAP) [9]. Here and in the following $\Gamma^{(n)}$ denotes the n-loop order, regularized and (minimally) subtracted, one-particle-irreducible (1PI) function. Note that the STIs and the WTIs are not able to fix the Green functions completely. Indeed it is possible to add invariant local terms to the action, changing the normalization conditions of the functions. A complete analysis of the normalization conditions for the SM can be found, for instance, in [10]–[14].

The application of a Taylor subtraction of the form $(1-T^\delta)$ on eqs. (1.1) and (1.2) transforms them into

$$\left[\mathcal{S}\left(\hat{\Gamma}\right)\right]^{(n)} = \hbar^n \Psi_S^{(n)} + \mathcal{O}(\hbar^{n+1}) \tag{1.3}$$

and

$$\mathcal{W}_{(\lambda)}\left(\hat{\Gamma}^{(n)}\right) = \hbar^n \Psi_W^{(n)}(\lambda) + \mathcal{O}(\hbar^{n+1})\,, \tag{1.4}$$

where $\hat{\Gamma}^{(n)} = (1-T^{\delta'})\Gamma^{(n)}$. A precise definition of T^δ and $T^{\delta'}$ can be found in to [6]. We only want to mention that δ has to be chosen in such a way that $(1-T^\delta)\Delta_{S/W}^{(n)} = 0$ and δ' corresponds to the power counting degree of the Green functions $\Gamma^{(n)}$. The new breaking terms $\Psi_S^{(n)}$ and $\Psi_W^{(n)}(\lambda)$ are due to over-subtractions and can be expressed in terms of a linear combination of ultraviolet (UV) finite Green functions and, thus, are independent of the regularization scheme [6]. Here we assumed that up to the $(n-1)$-loop order the Green functions are already renormalized correctly. The main difference between $\Psi_S^{(n)}$ and $\Psi_W^{(n)}(\lambda)$ is due to the linearity of the corresponding operators $\mathcal{S}$ and $\mathcal{W}_{(\lambda)}$, respectively. In the former case one has to consider non-linear terms arising from lower-loop orders. On the contrary in the latter the linearity of the WTI simplifies the evaluation of the breaking terms and counterterms enormously. Finally we introduce

$$\mathrm{I\!\Gamma}^{(n)} = \hat{\Gamma}^{(n)} + \Xi^{(n)} = (1-T^{\delta'})\Gamma^{(n)} + \Xi^{(n)}\,, \tag{1.5}$$

where the counterterm $\Xi^{(n)}$ is chosen in such a way that the following identities are fulfilled:

$$\left[\mathcal{S}\left(\mathrm{I\!\Gamma}\right)\right]^{(n)} = 0\,, \qquad \mathcal{W}_{(\lambda)}\left(\mathrm{I\!\Gamma}^{(n)}\right) = 0\,. \tag{1.6}$$

In general it is quite simple to compute the total counterterm $(-T^{\delta'}\Gamma^{(n)} + \Xi^{(n)})$, as it can be expressed in terms of Green functions expanded around zero external momenta.

As already mentioned above, there is still the freedom to add invariant counterterms, $\Xi_N^{(n)}$, to $\mathrm{I\!\Gamma}^{(n)}$ in eq. (1.5). In other words, we have the freedom to impose normalization conditions, which lead, in addition to eqs. (1.6), to the equations

$$\mathcal{N}_i\left(\mathrm{I\!\Gamma}^{(n)}\right) = 0\,, \tag{1.7}$$

where the index i runs over all independent parameters of the SM. As the Green function $\Gamma^{(n)}$ also has to fulfill this condition, we have for the counterterm

$$\mathcal{N}_i\left(-T^\delta\Gamma^{(n)} + \Xi^{(n)} + \Xi_N^{(n)}\right) = 0\,, \tag{1.8}$$

which is a local equation. This means that, whenever the effort to impose the normalization conditions is made, the changes due to the subtraction are only local changes, which can be easily compensated. Explicit examples will be discussed at the end of section 2. Notice that the imposition of normalization conditions is a very important ingredient of the computation in order to compare with other schemes and in order to simplify the breaking terms themselves.

The procedure described so far is based on the Taylor operator T^δ. In the presence of massless particles, this may introduce IR divergences. In the examples discussed in this paper, eventual IR problems are encountered in intermediate steps after neglecting one of the quark masses, a well-justified approximation in the case of the top-bottom doublet. Note that this kind of IR divergences should not be confused with those arising in connection with on-shell conditions of internal particle propagators. The appropriate methods for dealing with IR divergences are introduced in section 6.

Although we are mainly interested in the three-point functions, also some two-point functions with external (background) gauge bosons have to be treated properly in order to be able to renormalize the amplitudes correctly. They will be discussed in section 4.

In section 2 the one-loop sub-diagrams occurring in the two-loop calculation are analyzed. In section 3 the vertices involving only neutral gauge bosons are considered and, after introducing the two-point functions, we are ready to discuss the cases AWW and ZWW in section 5.

2. One-loop Green functions

This section is devoted to the one-loop sub-diagrams induced by QCD, which are needed for the renormalization.

In the case of neutral gauge bosons one has to take into account the two- and three-point Green functions[1] $\Gamma^{(1)}_{\bar{q}q}(p)$ and $\Gamma^{(1)}_{\hat{A}_\mu\bar{q}q}(p,q)$ and the corresponding vertices where the photon is replaced by the Z boson. For q we have $q \in \{u,d\}$, where u and d refer to a generic quark doublet. After the analysis of the WTIs, also the vertices with the neutral Goldstone boson, $\hat{G}^0$, $\Gamma^{(1)}_{\hat{G}^0\bar{u}u}(p,q)$ and $\Gamma^{(1)}_{\hat{G}^0\bar{d}d}(p,q)$, turn out to be relevant. For the amplitudes AWW and ZWW the vertices $\Gamma^{(1)}_{\hat{W}^+_\mu\bar{u}d}(p,q)$ and $\Gamma^{(1)}_{\hat{G}^+\bar{u}d}(p,q)$ are needed in addition. As we work in the framework of the BFM, no Green functions with external scalar or gauge fields have to be considered, and we

[1]All momenta are considered as incoming. In the Green functions $\Gamma_{\phi_1\cdots\phi_n}$ they are assigned to the corresponding fields starting from the right. The momentum of the most left field is determined via momentum conservation.

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are left with only three WTIs:

$$
\begin{aligned}
& i\,(p+q)^\mu \Gamma^{(1)}_{\hat{A}_\mu \bar{q} q}(p,q) + ie\,Q_q \left[\Gamma^{(1)}_{\bar{q}q}(q) - \Gamma^{(1)}_{\bar{q}q}(-p)\right] = \Delta^{(1)}_{W,\lambda_A \bar{q} q}(p,q)\,, \\
& i\,(p+q)^\mu \Gamma^{(1)}_{\hat{Z}_\mu \bar{q} q}(p,q) - M_Z \Gamma^{(1)}_{\hat{G}^0 \bar{q} q}(p,q) - \\
& -\; ie \left[(v_q - a_q \gamma_5)\Gamma^{(1)}_{\bar{q}q}(q) - \Gamma^{(1)}_{\bar{q}q}(-p)(v_q + a_q\gamma_5)\right] = \Delta^{(1)}_{W,\lambda_Z \bar{q} q}(p,q)\,, \\
& i\,(p+q)_\nu\, \Gamma^{(1)}_{\hat{W}^+_\nu \bar{u} d}(p,q) + i\,M_W \Gamma^{(1)}_{\hat{G}^+ \bar{u} d}(p,q) + \\
& \quad +\; \frac{ie}{s_W\sqrt{2}} \left[\Gamma^{(1)}_{\bar{u}u'}(-p) V_{u'd} P_L - V_{ud'} P_R \Gamma^{(1)}_{\bar{d}'d}(q)\right] = \Delta^{(1)}_{W,\lambda_+ \bar{u} d}(p,q)\,.
\end{aligned} \tag{2.1}
$$

Here and in the following we define the Weinberg angle through $c_W = M_W/M_Z$ as we want to maintain the form of the WTIs to be the same to all orders. The couplings of the fermions to the Z boson are given by $v_q = (I^3_q - 2s^2_W Q_q)/(2c_W s_W)$ and $a_q = I^3_q/(2c_W s_W)$, where I^3_q and Q_q are the third generator of $\mathrm{SU}_W(2)$ and the electric charge of the q quark, respectively. The equation for $\Delta_{W,\lambda_- \bar{d} u}$ has been omitted as it can easily be obtained from the last one in (2.1). $V_{qq'}$ are the CKM matrix elements where the summation over the primed quantities is understood and $P_{L/R} = (1 \mp \gamma_5)/2$ are the chiral projectors.

In order to remove the breaking terms we apply the Taylor operator $(1 - T^1_{p,q})$ as the Δ's have mass dimension one. This leads to

$$
\begin{aligned}
& i(p+q)^\mu \hat{\Gamma}^{(1)}_{\hat{A}_\mu \bar{q} q}(p,q) + ieQ_q \left[\hat{\Gamma}^{(1)}_{\bar{q}q}(q) - \hat{\Gamma}^{(1)}_{\bar{q}q}(-p)\right] = 0\,, \\
& i(p+q)^\mu \hat{\Gamma}^{(1)}_{\hat{Z}_\mu \bar{q} q}(p,q) - M_Z \hat{\Gamma}^{(1)}_{\hat{G}^0 \bar{q} q}(p,q) - \\
& - ie \left[(v_q - a_q \gamma_5)\hat{\Gamma}^{(1)}_{\bar{q}q}(q) - \hat{\Gamma}^{(1)}_{\bar{q}q}(-p)(v_q + a_q\gamma_5)\right] = \Psi^{(1)}_{W,\lambda_Z \bar{q} q}(p,q)\,, \\
& i\,(p+q)_\nu\, \hat{\Gamma}^{(1)}_{\hat{W}^+_\nu \bar{u} d}(p,q) + i\,M_W \hat{\Gamma}^{(1)}_{\hat{G}^+ \bar{u} d}(p,q) + \\
& \quad + \frac{ie}{s_W\sqrt{2}} \left[\hat{\Gamma}^{(1)}_{\bar{u}u'}(-p) V_{u'd} P_L - V_{ud'} P_R \hat{\Gamma}^{(1)}_{\bar{d}'d}(q)\right] = \Psi^{(1)}_{W,\lambda_+ \bar{u} d}(p,q)\,.
\end{aligned} \tag{2.2}
$$

The Ψ's, which occur by over-subtraction, are finite and read:

$$
\begin{aligned}
\Psi^{(1)}_{W,\lambda_Z \bar{q} q}(p,q) &= -M_Z \left(p^\rho \partial_{p^\rho} + q^\rho \partial_{q^\rho}\right) \Gamma^{(1)}_{\hat{G}^0 \bar{q} q}(p,q)\Big|_{p=q=0} \\
&= i\frac{\alpha_s}{4\pi} C_F \frac{e}{s_W c_W} I^3_q \left(\frac{1}{2}\xi - 2\right)(\not{p} + \not{q})\gamma_5\,, \\
\Psi^{(1)}_{W,\lambda_+ \bar{u} d}(p,q) &= i\,M_W \left(p^\rho \partial_{p^\rho} + q^\rho \partial_{q^\rho}\right) \Gamma^{(1)}_{\hat{G}^+ \bar{u} d}(p,q)\Big|_{p=q=0} \\
&= i\frac{\alpha_s}{4\pi} C_F \frac{eV_{ud}}{2\sqrt{2}s_W} \Bigg[\left(1 + \frac{1}{2}\xi\right)(\not{p} + \not{q}) P_L + \frac{\left(1 - \frac{1}{2}\xi\right)\ln\frac{m_u^2}{m_d^2}}{m_u^2 - m_d^2} \times \\
&\qquad \times \left(\left(m_u^2\,\not{p} + m_d^2\,\not{q}\right) P_L - 2m_u m_d (\not{p} + \not{q}) P_R\right)\Bigg]\,,
\end{aligned} \tag{2.3}
$$

where the gauge parameter ξ is defined through the gluon propagator $D_g(q) = i(-g^{\mu\nu} + \xi q^\mu q^\nu/q^2)/(q^2 + i\epsilon)$ and $C_F = 4/3$ is the colour factor. Owing to the linear momentum dependence of $\Psi^{(1)}_{W,\lambda_Z \bar{q}q}(p,q)$ and $\Psi^{(1)}_{W,\lambda_+ \bar{u}d}(p,q)$, their contribution can be absorbed by the counterterms

$$\begin{aligned} \Xi^{(1),W}_{\hat{Z}_\mu \bar{q}q}(p,q) &= \xi^{(1)}_{V\bar{q}q}\gamma^\mu + \xi^{(1)}_{A\bar{q}q}\gamma^\mu\gamma_5 \,, \\ \Xi^{(1),W}_{\hat{W}^+_\mu \bar{u}d}(p,q) &= \xi^{(1)}_{L,\hat{W}^+\bar{u}d}\gamma^\mu P_L + \xi^{(1)}_{R,\hat{W}^+\bar{u}d}\gamma^\mu P_R \,. \end{aligned} \tag{2.4}$$

for three point functions and

$$\Xi^{(1),N}_{\bar{q}q}(p) = \xi^{(1)}_{2,q}\,(\not{p} - m_q) + \xi^{(1)}_q m_q \,, \tag{2.5}$$

for quark self-energies. The coefficients $\xi^{(1)}_{2,q}$ and $\xi^{(1)}_q$ have to be tuned for the two-point function

$$\mathrm{I\!\Gamma}^{(1)}_{\bar{q}q}(p) = \Gamma^{(1)}_{\bar{q}q}(p) - \left[T^1_p \Gamma^{(1)}_{\bar{q}q}(p)\right] + \Xi^{(1),N}_{\bar{q}q}(p)\,, \tag{2.6}$$

to restore the WTI (2.2) and to satisfy the specific normalization conditions. In the case of the on-shell scheme, the condition $\mathrm{I\!\Gamma}^{(1)}_{\bar{q}q}(p^*) = 0$, for instance, where the real part of p^* corresponds to the physical quark mass, would fix $\xi^{(1)}_q$.

From the explicit results in eq. (2.3) one can see that the vector coefficient of $\Xi^{(1),W}_{\hat{Z}_\mu \bar{q}q}$ has to be zero. Concerning the axial-vector part, there are in principle two structures. However, since only the combination $(\not{p} + \not{q})$ appears in eq. (2.3) for $\Psi^{(1)}_{W,\lambda_Z \bar{q}q}(p,q)$, $\xi^{(1)}_{A\bar{q}q}$ is sufficient to remove the breaking term and we thus have

$$\xi^{(1)}_{V\bar{q}q} = 0\,, \qquad \xi^{(1)}_{A\bar{q}q} = \frac{\alpha_s}{4\pi} C_F \frac{e}{s_W c_W} I^3_q \left(\frac{1}{2}\xi - 2\right)\,. \tag{2.7}$$

Similarly we get in the case of $\Xi^{(1),W}_{\hat{W}^+_\mu \bar{u}d}(p,q)$

$$\begin{aligned} \xi^{(1)}_{2,u} &= \xi^{(1)}_{2,d} - \frac{\alpha_s}{8\pi} C_F \left(1 - \frac{1}{2}\xi\right) \ln\left(\frac{m_u^2}{m_d^2}\right), \\ \xi^{(1)}_{L,\hat{W}^+\bar{u}d} &= \frac{eV_{ud}}{2\sqrt{2}s_W}\left(\xi^{(1)}_{2,u} - \xi^{(1)}_{2,d}\right) + \frac{\alpha_s}{4\pi} C_F \frac{eV_{ud}}{2\sqrt{2}s_W}\left[\left(1 + \frac{1}{2}\xi\right) + \frac{\left(1 - \frac{1}{2}\xi\right) m_u^2 \ln\frac{m_u^2}{m_d^2}}{m_u^2 - m_d^2}\right], \\ \xi^{(1)}_{R,\hat{W}^+\bar{u}d} &= -\frac{\alpha_s}{4\pi} C_F \frac{eV_{ud}}{\sqrt{2}s_W} \frac{m_u m_d \left(1 - \frac{1}{2}\xi\right) \ln\frac{m_u^2}{m_d^2}}{m_u^2 - m_d^2}\,. \end{aligned} \tag{2.8}$$

The free parameter $\xi^{(1)}_{2,d}$ can be fixed by a normalization condition on the two-point function residue.

Finally we can write down the symmetric one-loop Green functions for the neutral and charged current vertices:

$$
\begin{aligned}
\mathbb{I}\Gamma^{(1)}_{\hat{A}_\mu \bar{q} q}(p,q) &= \Gamma^{(1)}_{\hat{A}_\mu \bar{q} q}(p,q) - \left[T^0_{p,q}\Gamma^{(1)}_{\hat{A}_\mu \bar{q} q}(p,q)\right] - eQ_q\xi^{(1)}_{2,q}\gamma^\mu \,, \\
\mathbb{I}\Gamma^{(1)}_{\hat{Z}_\mu \bar{q} q}(p,q) &= \Gamma^{(1)}_{\hat{Z}_\mu \bar{q} q}(p,q) - \left[T^0_{p,q}\Gamma^{(1)}_{\hat{Z}_\mu \bar{q} q}(p,q) - \xi^{(1)}_{A\bar{q}q}\gamma^\mu\gamma_5\right] + e\xi^{(1)}_{2,q}\gamma^\mu\left(v_q - a_q\gamma_5\right) , \\
\mathbb{I}\Gamma^{(1)}_{\hat{G}^0 \bar{q} q}(p,q) &= \Gamma^{(1)}_{\hat{G}^0 \bar{q} q}(p,q) - \left[T^0_{p,q}\Gamma^{(1)}_{\hat{G}^0 \bar{q} q}(p,q)\right] - \frac{eI^3_q m_q}{is_W M_W}\left(\xi^{(1)}_q - \xi^{(1)}_{2,q}\right)\gamma^5 \,, \\
\mathbb{I}\Gamma^{(1)}_{\hat{W}^+_\mu \bar{u} d}(p,q) &= \Gamma^{(1)}_{\hat{W}^+_\mu \bar{u} d}(p,q) - \left[T^0_{p,q}\Gamma^{(1)}_{\hat{W}^+_\mu \bar{u} d}(p,q) - \xi^{(1)}_{L,\hat{W}^+\bar{u}d}\gamma^\mu P_L - \xi^{(1)}_{R,\hat{W}^+\bar{u}d}\gamma^\mu P_R\right] + \\
&\quad + \frac{eV_{ud}}{2\sqrt{2}s_W}\left(\xi^{(1)}_{2,u} + \xi^{(1)}_{2,d}\right)\gamma^\mu P_L \,, \\
\mathbb{I}\Gamma^{(1)}_{\hat{G}^+ \bar{u} d}(p,q) &= \Gamma^{(1)}_{\hat{G}^+ \bar{u} d}(p,q) - \left[T^0_{p,q}\Gamma^{(1)}_{\hat{G}^+ \bar{u} d}(p,q)\right] - \frac{eV_{ud}m_u}{\sqrt{2}s_W M_W}\left(\xi^{(1)}_u - \xi^{(1)}_{2,u}\right)P_L + \\
&\quad + \frac{eV_{ud}m_d}{\sqrt{2}s_W M_W}\left(\xi^{(1)}_d - \xi^{(1)}_{2,d}\right)P_R \,.
\end{aligned}
\tag{2.9}
$$

In this specific sector only the counterterms $\xi^{(1)}_u$, $\xi^{(1)}_d$ and $\xi^{(1)}_{2,d}$ can be tuned to imposed suitable normalization conditons, the others are indeed necessary to restore the WTI (2.2).

Using the notation of the introduction, eqs. (2.9) can be expressed in the following compact form

$$
\begin{aligned}
\mathbb{I}\Gamma^{(n)} &= \hat{\Gamma}^{(n)} + \Xi^{(n)} \\
&= \Gamma^{(n)} - \left[T^\delta\Gamma^{(n)} + \Xi^{(n)}_W\right] + \Xi^{(n)}_N \\
&= \Gamma^{(n)}_{\text{bare}} - \Gamma^{(n)}_{\text{UV}} - \left[T^\delta\Gamma^{(n)}_{\text{bare}} - T^\delta\Gamma^{(n)}_{\text{UV}} + \Xi^{(n)}_W\right] + \Xi^{(n)}_N \,,
\end{aligned}
\tag{2.10}
$$

In the third line we have introduced the bare Green function $\Gamma^{(n)}_{\text{bare}}$. This quantity is defined by $\Gamma^{(n)} = \Gamma^{(n)}_{\text{bare}} - \Gamma^{(n)}_{\text{UV}}$, where $\Gamma^{(n)}_{\text{UV}}$ denotes the necessary UV counterterms computed in the specified regularization. Clearly, the complete one-loop counterterms, namely $\mathbb{I}\Gamma^{(n)} - \Gamma^{(n)}_{\text{bare}}$, have to be taken into account at the two-loop level.

3. Neutral-gauge-boson vertices

3.1 The AZZ case

The vertex involving a photon and two Z bosons is used to demonstrate the main features of our technique. This example clarifies also the issue of anomaly cancellation in our formalism. In principle there is also the vertex ZAA. However, it is very similar to AZZ. Thus we will not present explicit results for ZAA as the corresponding equations are simply obtained by replacing one of the Z bosons by a photon.

As we are dealing with external background fields, WTIs can be used to fix the counterterms for these amplitudes. In order to derive the complete set of identities, one of the gauge fields has to be replaced by the infinitesimal parameter of the background gauge transformations and then the derivatives of the functional WTI have to be performed (cf. ref. [6]). This leads to six identities, which naturally split into two sets depending on whether the index of the photon or the Z boson is contracted with the external momentum. We get the following closed (under renormalization) set of equations

$$\begin{aligned} i(p+q)^\mu \Gamma^{(n)}_{\hat{A}_\mu \hat{Z}_\nu \hat{Z}_\rho}(p,q) &= \Delta^{(n)}_{W,\lambda_A \hat{Z}_\nu \hat{Z}_\rho}(p,q)\,, \\ -ip^\nu \Gamma^{(n)}_{\hat{A}_\mu \hat{Z}_\nu \hat{Z}_\rho}(p,q) - M_Z \Gamma^{(n)}_{\hat{A}_\mu \hat{G}^0 \hat{Z}_\rho}(p,q) &= \Delta^{(n)}_{W,\hat{A}_\mu \lambda_Z \hat{Z}_\rho}(p,q)\,, \\ i(p+q)^\mu \Gamma^{(n)}_{\hat{A}_\mu \hat{G}^0 \hat{Z}_\rho}(p,q) &= \Delta^{(n)}_{W,\lambda_A \hat{G}^0 \hat{Z}_\rho}(p,q)\,, \\ -ip^\nu \Gamma^{(n)}_{\hat{A}_\mu \hat{Z}_\nu \hat{G}^0}(p,q) - M_Z \Gamma^{(n)}_{\hat{A}_\mu \hat{G}^0 \hat{G}^0}(p,q) &= \Delta^{(n)}_{W,\hat{A}_\mu \lambda_Z \hat{G}^0}(p,q)\,, \\ i(p+q)^\mu \Gamma^{(n)}_{\hat{A}_\mu \hat{G}^0 \hat{G}^0}(p,q) &= \Delta^{(n)}_{W,\lambda_A \hat{G}^0 \hat{G}^0}(p,q)\,. \end{aligned} \tag{3.1}$$

There are in principle three more equations where the contraction is performed with q^ρ. However, they contain no new information.

The breaking terms in eqs. (3.1) have mass dimension two. Thus we have to apply the operator $(1 - T^2_{p,q})$ in order to remove them. This leads to

$$i(p+q)^\mu \hat{\Gamma}^{(n)}_{\hat{A}_\mu \hat{Z}_\nu \hat{Z}_\rho}(p,q) = \Psi^{(n)}_{W,\lambda_A \hat{Z}_\nu \hat{Z}_\rho}(p,q) = 0\,, \tag{3.2}$$

$$-ip^\nu \hat{\Gamma}^{(n)}_{\hat{A}_\mu \hat{Z}_\nu \hat{Z}_\rho}(p,q) - M_Z \hat{\Gamma}^{(n)}_{\hat{A}_\mu \hat{G}^0 \hat{Z}_\rho}(p,q) = \Psi^{(n)}_{W,\hat{A}_\mu \lambda_Z \hat{Z}_\rho}(p,q)\,, \tag{3.3}$$

$$i(p+q)^\mu \hat{\Gamma}^{(n)}_{\hat{A}_\mu \hat{G}^0 \hat{Z}_\rho}(p,q) = \Psi^{(n)}_{W,\lambda_A \hat{G}^0 \hat{Z}_\rho}(p,q) = 0\,, \tag{3.4}$$

$$-ip^\nu \hat{\Gamma}^{(n)}_{\hat{A}_\mu \hat{Z}_\nu \hat{G}^0}(p,q) - M_Z \hat{\Gamma}^{(n)}_{\hat{A}_\mu \hat{G}^0 \hat{G}^0}(p,q) = \Psi^{(n)}_{W,\hat{A}_\mu \lambda_Z \hat{G}^0}(p,q) = 0\,, \tag{3.5}$$

$$i(p+q)^\mu \hat{\Gamma}^{(n)}_{\hat{A}_\mu \hat{G}^0 \hat{G}^0}(p,q) = \Psi^{(n)}_{W,\lambda_A \hat{G}^0 \hat{G}^0}(p,q) = 0\,, \tag{3.6}$$

where only $\Psi^{(n)}_{W,\hat{A}_\mu \lambda_Z \hat{Z}_\rho}(p,q)$ is non vanishing:

$$\begin{aligned} \Psi^{(n)}_{W,\hat{A}_\mu \lambda_Z \hat{Z}_\rho}(p,q) = &-\frac{M_Z}{2} \left(p^\alpha p^\beta \partial_{p^\alpha} \partial_{p^\beta} + 2 p^\alpha q^\beta \partial_{p^\alpha} \partial_{q^\beta} + q^\alpha q^\beta \partial_{q^\alpha} \partial_{q^\beta} \right) \times \\ &\times \hat{\Gamma}^{(n)}_{\hat{A}_\mu \hat{G}^0 \hat{Z}_\rho}(p,q) \Big|_{p=q=0}\,. \end{aligned} \tag{3.7}$$

The other breaking terms are zero because of QED-like WTIs for the external background photon (eqs. (3.2), (3.4) and (3.6)) and Lorentz invariance (eq. (3.5)), respectively.

In order to remove $\Psi^{(n)}_{W,\hat{A}_\mu \lambda_Z \hat{Z}_\rho}(p,q)$, a counterterm, $\Xi^{(n)}_{\hat{A}_\mu \hat{Z}_\nu \hat{Z}_\rho}$, has to be introduced for the Green functions $\hat{\Gamma}^{(n)}_{\hat{A}_\mu \hat{Z}_\nu \hat{Z}_\rho}(p,q)$. Notice, however, that this Green function also appears in eq. (3.2). In order not to spoil eq. (3.2), $\Xi^{(n)}_{\hat{A}_\mu \hat{Z}_\nu \hat{Z}_\rho}$ has to be longitudinal w.r.t. the photon index μ. On the other hand, if we contract eq. (3.3) by $(p+q)^\mu$ and use eqs. (3.2) and (3.4), we obtain

$$(p+q)^\mu \Psi^{(n)}_{W,\hat{A}_\mu \lambda_Z \hat{Z}_\rho}(p,q) = 0 \,. \tag{3.8}$$

This implies that the breaking term $\Psi^{(n)}_{W,\hat{A}_\mu \lambda_Z \hat{Z}_\rho}(p,q)$ should be transversal w.r.t. the photon index μ. Thus, combining the two arguments, we deduce that the breaking term itself has to be zero.

We have checked this prediction by explicit calculations at the one- and two-loop levels. At one-loop order, the contribution from one fermion species gives

$$\Psi^{(1)}_{W,\hat{A}_\mu \lambda_Z \hat{Z}_\rho}(p,q) = i\frac{\alpha}{4\pi}\frac{4e}{s_W c_W} I_3^q Q_q v_q \epsilon^{\mu\rho\alpha\beta} p_\alpha q_\beta \,, \tag{3.9}$$

with $\epsilon^{0123} = 1$. The only reminder on the fermion type is the third component of the isospin, the charge and the coupling to the Z boson. Thus, after summing over a complete family of quarks and leptons one gets zero. This is the same mechanism which leads to the cancellation of the Adler-Bardeen anomaly in the SM [15]. At two loops already the sum over all contributing diagrams of one quark flavour is zero as we checked by an explicit calculation.

This example provides a nice demonstration of the power of our technique. Regardless of the regularization adopted to compute the Green functions $\Gamma^{(n)}$, the zero-momentum subtraction fixes automatically the non-invariant counterterms needed to restore the symmetries. In particular, we found that besides one-loop counterterms (which were discussed in section 2) no other counterterm is necessary to define the properly renormalized amplitudes. Finally, the direct computation of breaking term $\Psi^{(n)}_{W,\hat{A}_\mu \lambda_Z \hat{Z}_\rho}(p,q)$ at one- and two-loop level ($n = 1,2$) shows how the anomaly coefficient can be computed and the Adler-Bardeen non-renormalization theorem can be verified in the present framework.

3.2 The ZZZ case

Also in the ZZZ case there is no tree-level contribution, which makes it similar to the previous case. Here the closed system of WTIs looks as follows:

$$\begin{aligned}
i(p+q)^\mu \Gamma^{(n)}_{\hat{Z}_\mu \hat{Z}_\nu \hat{Z}_\rho}(p,q) - M_Z \Gamma^{(n)}_{\hat{G}^0 \hat{Z}_\nu \hat{Z}_\rho}(p,q) &= \Delta^{(n)}_{W,\lambda_Z \hat{Z}_\nu \hat{Z}_\rho}(p,q) \,, \\
i(p+q)^\mu \Gamma^{(n)}_{\hat{Z}_\mu \hat{G}^0 \hat{Z}_\rho}(p,q) - M_Z \Gamma^{(n)}_{\hat{G}^0 \hat{G}^0 \hat{Z}_\rho}(p,q) &= \Delta^{(n)}_{W,\lambda_Z \hat{G}^0 \hat{Z}_\rho}(p,q) \,, \\
i(p+q)^\mu \Gamma^{(n)}_{\hat{Z}_\mu \hat{G}^0 \hat{G}}(p,q) - M_Z \Gamma^{(n)}_{\hat{G}^0 \hat{G}^0 \hat{G}^0}(p,q) &= \Delta^{(n)}_{W,\lambda_Z \hat{G}^0 \hat{G}^0}(p,q) \,.
\end{aligned} \tag{3.10}$$

Taylor subtraction, by application of the operator $(1 - T^2_{p,q})$, partially eliminates the breaking terms and leads to

$$\Psi^{(n)}_{W,\lambda_Z\hat{Z}_\nu\hat{Z}_\rho}(p,q) = -\frac{M_Z}{2}\Big[\left(p^\alpha p^\beta \partial_{p^\alpha}\partial_{p^\beta} + 2p^\alpha q^\beta \partial_{p^\alpha}\partial_{q^\beta} + q^\alpha q^\beta \partial_{q^\alpha}\partial_{q^\beta}\right) \times \times \hat{\Gamma}^{(n)}_{\hat{G}^0\hat{Z}_\nu\hat{Z}_\rho}(p,q)\Big]_{p=q=0}, \tag{3.11}$$

$$\Psi^{(n)}_{W,\lambda_Z\hat{G}^0\hat{Z}_\rho}(p,q) = 0\,,$$

$$\Psi^{(n)}_{W,\lambda_Z\hat{G}^0\hat{G}^0}(p,q) = -\frac{M_Z}{2}\Big[\left(p^\alpha p^\beta \partial_{p^\alpha}\partial_{p^\beta} + 2p^\alpha q^\beta \partial_{p^\alpha}\partial_{q^\beta} + q^\alpha q^\beta \partial_{q^\alpha}\partial_{q^\beta}\right) \times \times \hat{\Gamma}^{(n)}_{\hat{G}^0\hat{G}^0\hat{G}^0}(p,q)\Big]_{p=q=0}. \tag{3.12}$$

Owing to CP invariance and the fact that we consider only QCD corrections, it turns out that both $\Psi^{(n)}_{W,\lambda_Z\hat{Z}_\nu\hat{Z}_\rho}(p,q)$ and $\Psi^{(n)}_{W,\lambda_Z\hat{G}^0\hat{G}^0}(p,q)$ are zero at the one- and two-loop order, i.e. for $n=1,2$. This has been checked by explicit one- and two-loop calculations. Actually $\Psi^{(n)}_{W,\lambda_Z\hat{Z}_\nu\hat{Z}_\rho}(p,q)$ shows the same behaviour as the corresponding contribution in the AZZ case: for $n=1$ the breaking term vanishes after summing over a whole family and at two-loop order the contribution from each fermion species gives separately zero.

At higher orders, CP violation induced by the CKM quark mixing matrix might render $\Psi^{(n)}_{W,\lambda_Z\hat{Z}_\nu\hat{Z}_\rho}(p,q)$ and $\Psi^{(n)}_{W,\lambda_Z\hat{G}^0\hat{G}^0}(p,q)$ different from zero. The corresponding counterterm for $\Gamma^{(n)}_{\hat{Z}_\mu\hat{Z}_\nu\hat{Z}_\rho}$, which has to be introduced in order to remove $\Psi^{(n)}_{W,\lambda_Z\hat{Z}_\nu\hat{Z}_\rho}$ (p,q), would read

$$\Xi^{(n),W}_{\hat{Z}_\mu\hat{Z}_\nu\hat{Z}_\rho}(p,q) = \xi^{(n)}_{\hat{Z}\hat{Z}\hat{Z}}\,\left(p^\nu g^{\mu\rho} + q^\rho g^{\mu\nu} - (p+q)^\mu g^{\nu\rho}\right), \tag{3.13}$$

where the coefficient $\xi^{(n)}_{\hat{Z}\hat{Z}\hat{Z}}$ is determined in terms of the breaking terms (3.11). Notice that, in the present case, there is no condition like eq. (3.8) and thus, unlike the AZZ case, in general there are counterterms of the form (3.13).

4. Fermionic contribution to the two-point functions

A proper renormalization of the two-point functions is needed in order to be able to correctly renormalize the three-point Green functions AWW and ZWW. Moreover, most of the normalization conditions are expressed in terms of two-point functions. In this section we mainly concentrate on the results needed in the forthcoming parts of the paper, while details can be found in ref. [6].

Applying our prescriptions to the two-point Green functions with external background fields shows that only the self-energies of the (background) W and Z bosons are affected by breaking terms. The corresponding symmetrical Green functions read

$$\begin{aligned}\mathrm{I\Gamma}^{(n)}_{\hat{W}^+_\mu\hat{W}^-_\nu}(p) &= \Gamma^{(n)}_{\hat{W}^+_\mu\hat{W}^-_\nu}(p) - \Big[T^2_p\Gamma^{(n)}_{\hat{W}^+_\mu\hat{W}^-_\nu}(p) + \xi^{(n)}_{\hat{W},1}\,p^2 g_{\mu\nu} + \xi^{(n)}_{\hat{W},2}\,p_\mu p_\nu\Big] + \xi^{(n)}_{M_W} g_{\mu\nu}\,,\\ \mathrm{I\Gamma}^{(n)}_{\hat{Z}_\mu\hat{Z}_\nu}(p) &= \Gamma^{(n)}_{\hat{Z}_\mu\hat{Z}_\nu}(p) - \Big[T^2_p\Gamma^{(n)}_{\hat{Z}_\mu\hat{Z}_\nu}(p) + \xi^{(n)}_{\hat{Z},1}\,p^2 g_{\mu\nu} + \xi^{(n)}_{\hat{Z},2}\,p_\mu p_\nu\Big] + \xi^{(n)}_{M_Z} g_{\mu\nu}\,,\end{aligned} \tag{4.1}$$

where the functions $\xi^{(n)}_{\hat V,i}$ $(V=W,Z; i=1,2)$ are obtained from the following expressions:

$$
\begin{aligned}
\xi^{(n)}_{\hat W,1}+\xi^{(n)}_{\hat W,2} &= \frac{M_W}{2}\,\partial_p^2\partial_{p_\rho}\Gamma^{(n)}_{\hat G^+\hat W^-_\rho}(p)\Big|_{p=0}\,,\\
\xi^{(n)}_{\hat Z,1}+\xi^{(n)}_{\hat Z,2} &= i\frac{M_Z}{2}\,\partial_p^2\partial_{p_\rho}\Gamma^{(n)}_{\hat G^0\hat Z_\rho}(p)\Big|_{p=0}\,.
\end{aligned}
\tag{4.2}
$$

Equation (4.2) only fixes the sum of the coefficients $\xi^{(n)}_{\hat V,1}$ and $\xi^{(n)}_{\hat V,2}$. Note, however, that one still has the freedom of implementing suitable renormalization conditions for the two-point functions. For example, one can decide to renormalize the W- or the Z-background two-point functions on-shell. This fixes the difference of $\xi^{(i)}_{\hat V,1}$ and $\xi^{(i)}_{\hat V,2}$. However, as is well known [16, 17, 13], the wave function renormalization of the background fields is related to the coupling constant renormalization and, therefore, the WTIs for three-point functions, discussed in the next section, provide a proper renormalization of the difference.

The coefficients $\xi^{(n)}_{M_V}$ $(V=W/Z)$ are fixed via the normalization conditions. In the case of the on-shell scheme [10, 18], where the pole mass[2] enters as a parameter, they would look as follows $(i=1,2)$

$$
\begin{aligned}
\mathrm{I\!\Gamma}^{(n),T}_{\hat W^+\hat W^-}(p^*) &= 0 \qquad \text{with} \qquad \mathrm{Re}(p^*)^2 = M_W^2\,,\\
\det\begin{pmatrix}\mathrm{I\!\Gamma}^{(n),T}_{\hat Z\hat Z}(p^*) & \mathrm{I\!\Gamma}^{(n),T}_{\hat Z\hat A}(p^*)\\ \mathrm{I\!\Gamma}^{(n),T}_{\hat A\hat Z}(p^*) & \mathrm{I\!\Gamma}^{(n),T}_{\hat A\hat A}(p^*)\end{pmatrix} &= 0 \qquad \text{with} \qquad \mathrm{Re}(p^*)^2 = M_Z^2\,,
\end{aligned}
\tag{4.3}
$$

where the superscript T marks the transversal parts. In this framework M_W^2 and M_Z^2 are the physical masses, which also serve as input parameters.

Besides mass renormalizations, we have to take into account the renormalization of the photon self-energy and its mixing with the Z bosons. The structure of counterterms for the mixed two-point functions and for the photon two-point function can be found in the literature [12, 14, 13]. We can impose the following normalization conditions

$$
\mathrm{I\!\Gamma}^{(n),T}_{\hat A\hat A}(0)=0\,,\qquad \mathrm{I\!\Gamma}^{(n),T}_{\hat A\hat Z}(0)=0\,. \tag{4.4}
$$

At the end of this section we want to note that at one- and two-loop order the Green functions $\Gamma^{(n)}_{\hat W^+_\mu\hat W^-_\nu}(p)$ and $\Gamma^{(n)}_{\hat Z_\mu\hat Z_\nu}(p)$ in eq. (4.1) could be chosen to be already symmetric, since, in the case of the two-point functions, there exists an effectively invariant regularization. It can be shown that in this case the naive prescription of γ_5 accompanied with dimensional regularization leads to the correct answer (see, e.g. [19]). The symmetry is destroyed by Taylor subtraction and again restored by the counterterms which means that the quantities in (4.1) can be written as

$$
\mathrm{I\!\Gamma}_{\hat V_1\hat V_2}(p)=(1-T_p^2)\mathrm{I\!\Gamma}_{\hat V_1\hat V_2}(p)+\Xi_{\hat V_1\hat V_2}\,. \tag{4.5}
$$

[2]As was proven in [13] this is equivalent to the pole renormalization for the quantum gauge bosons.

Thus, in this case the method of algebraic renormalization is not needed. However, the calculation of the counterterms $\Xi_{\hat{V}_1\hat{V}_2}$ in eq. (4.5) is necessary as the renormalization of the three-point functions with the help of Taylor subtraction — which we present in the next section — depends on these counterterms.

5. Charged gauge boson vertices

5.1 The case AWW

The analysis of the vertex AWW turns out to be the most important for phenomenological studies, and it entails several interesting features. In particular, in contrast to the previous examples, AZZ and ZZZ, the WTIs for the AWW amplitudes appear more cumbersome because of the presence of two-point functions. In the conventional algebraic renormalization methods this is very cumbersome, as all Green functions appearing in the WTIs have to be computed — often for off-shell momenta. We will see that in our approach only a few Green functions remain which have to be evaluated with zero external momentum.

In order to obtain a closed set of identities we have to differentiate w.r.t. $\lambda_A \hat{W}^+_\rho \hat{W}^-_\sigma$ and $\hat{A}_\mu \lambda_+ \hat{W}^-_\sigma$. The first option leads to

$$
\begin{aligned}
& i\,(p_+ + p_-)^\mu\, \Gamma^{(n)}_{\hat{A}_\mu \hat{W}^+_\rho \hat{W}^-_\sigma}(p_+,p_-) - ie\left(\mathrm{I\!\Gamma}^{(n)}_{\hat{W}^+_\rho \hat{W}^-_\sigma}(p_-) - \mathrm{I\!\Gamma}^{(n)}_{\hat{W}^+_\rho \hat{W}^-_\sigma}(-p_+)\right) = \\
&\qquad = \Delta'^{,(n)}_{W,\lambda_A \hat{W}^+_\rho \hat{W}^-_\sigma}(p_+,p_-)\,, \\
& i\,(p_+ + p_-)^\mu\, \Gamma^{(n)}_{\hat{A}_\mu \hat{G}^+ \hat{G}^-}(p_+,p_-) - ie\left(\mathrm{I\!\Gamma}^{(n)}_{\hat{G}^+ \hat{G}^-}(p_-) - \mathrm{I\!\Gamma}^{(n)}_{\hat{G}^+ \hat{G}^-}(-p_+)\right) = \\
&\qquad = \Delta'^{,(n)}_{W,\lambda_A \hat{G}^+ \hat{G}^-}(p_+,p_-)\,, \\
& i\,(p_+ + p_-)^\mu\, \Gamma^{(n)}_{\hat{A}_\mu \hat{W}^+_\rho \hat{G}^-}(p_+,p_-) - ie\left(\mathrm{I\!\Gamma}^{(n)}_{\hat{W}^+_\rho \hat{G}^-}(p_-) - \mathrm{I\!\Gamma}^{(n)}_{\hat{W}^+_\rho \hat{G}^-}(-p_+)\right) = \\
&\qquad = \Delta'^{,(n)}_{W,\lambda_A \hat{W}^+_\rho \hat{G}^-}(p_+,p_-)\,.
\end{aligned} \tag{5.1}
$$

In analogy we obtain, from the functional differentiation w.r.t. $\hat{A}_\mu \lambda_+ \hat{W}^-_\sigma$:

$$
\begin{aligned}
& -ip_+^\rho \Gamma^{(n)}_{\hat{A}_\mu \hat{W}^+_\rho \hat{W}^-_\sigma}(p_+,p_-) + iM_W \Gamma^{(n)}_{\hat{A}_\mu \hat{G}^+ \hat{W}^-_\sigma}(p_+,p_-) + \\
& + ie\left(\mathrm{I\!\Gamma}^{(n)}_{\hat{W}^+_\mu \hat{W}^-_\sigma}(p_-) - \mathrm{I\!\Gamma}^{(n)}_{\hat{A}_\mu \hat{A}_\sigma}(-p_+ - p_-) + \frac{c_W}{s_W}\mathrm{I\!\Gamma}^{(n)}_{\hat{A}_\mu \hat{Z}_\sigma}(-p_+ - p_-)\right) = \\
&\qquad = \Delta'^{,(n)}_{W,A_\mu \lambda_+ \hat{W}^-_\sigma}(p_+,p_-)\,, \\
& -ip_+^\rho \Gamma^{(n)}_{\hat{A}_\mu \hat{W}^+_\rho \hat{G}^-}(p_+,p_-) + iM_W \Gamma^{(n)}_{\hat{A}_\mu \hat{G}^+ \hat{G}^-}(p_+,p_-) + \\
& + e\left(i\mathrm{I\!\Gamma}^{(n)}_{\hat{W}^+_\mu \hat{G}^-}(p_-) + \frac{1}{2s_W}\mathrm{I\!\Gamma}^{(n)}_{\hat{A}_\mu \hat{G}^0}(-p_+ - p_-) - \frac{i}{2s_W}\mathrm{I\!\Gamma}^{(n)}_{\hat{A}_\mu \hat{H}}(-p_+ - p_-)\right) = \\
&\qquad = \Delta'^{,(n)}_{W,A_\mu \lambda_+ \hat{G}^-}(p_+,p_-)\,.
\end{aligned} \tag{5.2}
$$

Again all equations that can be obtained by interchanging W^+ and W^- are omitted. In the above equations it is assumed that the two-point functions are already correctly renormalized according to (4.1). Therefore a prime is added to the corresponding Δ on the r.h.s..

By performing a Taylor subtraction $(1 - T^2_{p^+,p^-})$, the above equations lead to the following universal breaking terms, written in terms of the Taylor-subtracted Green functions $\hat{\Gamma}^{(n)}$:

$$\begin{aligned}
\Psi'^{,(n)}_{\lambda_A \hat{W}^+_\rho \hat{W}^-_\sigma}(p_+, p_-) &= -ie\left(T^2_{p^-}\,\mathrm{I\!\Gamma}^{(n)}_{\hat{W}^+_\rho \hat{W}^-_\sigma}(p_-) - T^2_{p^+}\,\mathrm{I\!\Gamma}^{(n)}_{\hat{W}^+_\rho \hat{W}^-_\sigma}(-p_+)\right)\\
&= ie\left[\left(\xi^{(n)}_{\hat{W},1}\,p_-^2 g_{\rho\sigma} + \xi^{(n)}_{\hat{W},2}\,p_{-,\rho}p_{-,\sigma}\right) - \right.\\
&\qquad \left. - \left(\xi^{(n)}_{\hat{W},1}\,p_+^2 g_{\rho\sigma} + \xi^{(n)}_{\hat{W},2}\,p_{+,\rho}p_{+,\sigma}\right)\right],\\
\Psi'^{,(n)}_{\lambda_A \hat{G}^+ \hat{G}^-}(p_+, p_-) &= \Psi'^{,(n)}_{\lambda_A \hat{W}^+_\rho \hat{G}^-}(p_+, p_-) = 0\,,\\
\Psi'^{,(n)}_{\hat{A}_\mu \lambda_+ \hat{W}^-_\sigma}(p_+, p_-) &= iM_W\left(T^2_{p^+,p^-} - T^1_{p^+,p^-}\right)\Gamma^{(n)}_{\hat{A}_\mu \hat{G}^+ \hat{W}^-_\sigma}(p_+, p_-) + ieT^2_{p^-}\,\mathrm{I\!\Gamma}^{(n)}_{\hat{W}^+_\mu \hat{W}^-_\sigma}(p_-)\\
&= \frac{i\,M_W}{2}\left(p_+^\rho p_+^\nu \partial_{p_+^\rho}\partial_{p_+^\nu} + 2\,p_+^\rho p_-^\nu \partial_{p_+^\rho}\partial_{p_-^\nu} + p_-^\rho p_-^\nu \partial_{p_-^\rho}\partial_{p_-^\nu}\right) \times\\
&\quad \times \Gamma^{(n)}_{\hat{A}_\mu \hat{G}^+ \hat{W}^-_\sigma}(p_+, p_-)\Big|_{p_\pm = 0} - ie\left(\xi^{(n)}_{\hat{W},1} p_-^2 g_{\mu\sigma} + \xi^{(n)}_{\hat{W},2} p_{-,\mu}p_{-,\sigma}\right),\\
\Psi'^{,(n)}_{\hat{A}_\mu \lambda_+ \hat{G}^-}(p_+, p_-) &= 0\,.
\end{aligned} \tag{5.3}$$

The r.h.s. of the last term in eq. (5.3) vanishes because of covariance and zero-momentum subtraction. Notice the appearance of the breaking terms for the W boson two-point functions. In principle also the corresponding ones from the charged Goldstone boson could appear. However, this can be avoided as zero momentum subtraction of the Goldstone self-energies automatically preserves the respective WTIs (cf. ref. [6]). Note that due to the fact that the photon is massless there is no contribution from $\mathrm{I\!\Gamma}^{(n)}_{\hat{A}_\mu \hat{H}}$. In our approximation it furthermore does not contribute due to CP violation.

In order to restore the WTIs one has the freedom of adding a non-invariant counterterm to the Green function $\Gamma_{\hat{A}_\mu \hat{W}^+_\rho \hat{W}^-_\sigma}$. However, to remove $\Psi^{(n)}_{A_\mu \lambda_+ \hat{W}^-_\sigma}(p_+, p_-)$ it is necessary that the latter is independent of $p_\pm^2\, g_{\mu\sigma}$ and $p_{\pm,\mu}p_{\pm,\sigma}$. This can be achieved by fixing the difference between the parameters $\xi^{(n)}_{\hat{W},1} - \xi^{(n)}_{\hat{W},2}$. Requiring that WTIs be preserved in the tree-level form amounts to imposing a charge renormalization.

In a next step we want to translate the information about the breaking terms into counterterms that will restore the symmetry of the Green functions. In general the QAP allows for all possible breaking terms with suitable dimensions. However, not all of them are independent. The consistency conditions can be used to figure out the independent ones and thus reduce the counterterms to a minimal set.

In the AWW case the most general counterterm that can be used to re-absorb the breaking term is of the form

$$\begin{aligned}\Xi^{(n)}_{\hat{A}_\mu \hat{W}^+_\nu \hat{W}^-_\rho}(p^+,p^-) = e\Big[& g_{\mu\nu}\left(\xi_1^{(n)} p^+_\rho + \xi_2^{(n)} p^-_\rho\right) + g_{\nu\rho}\left(\xi_3^{(n)} p^+_\mu + \xi_4^{(n)} p^-_\mu\right) + \\ & + g_{\mu\rho}\left(\xi_5^{(n)} p^+_\nu + \xi_6^{(n)} p^-_\nu\right)\Big]\,,\end{aligned} \tag{5.4}$$

where the coefficients $\xi_i^{(n)}$ can be expressed through the breaking terms as we will show in the following. Owing to the first equations of (5.1) and (5.3) one obtains

$$\begin{aligned}&\xi_1^{(n)} + \xi_6^{(n)} = 0\,, \qquad \xi_2^{(n)} + \xi_5^{(n)} = 0\,, \qquad -\xi_4^{(n)} = \xi_3^{(n)} = \xi^{(n)}_{\hat{W},1}\,, \\ &-(\xi_2^{(n)} + \xi_6^{(n)}) = \xi_1^{(n)} + \xi_5^{(n)} = \xi^{(n)}_{\hat{W},2}\,.\end{aligned} \tag{5.5}$$

Note, that the sum of $\xi^{(n)}_{\hat{W},1} + \xi^{(n)}_{\hat{W},2}$ is given in eq. (4.2). The contraction of (5.4) with p^ρ_+ and the comparison with $\Psi^{(n)}_{\hat{A}_\mu \lambda_+ \hat{W}^-_\sigma}$ in (5.3) leads to another set of equations. At first sight there are more equations than unknowns. However, one should have in mind that not all equations are independent due to WTIs like

$$\begin{aligned}&\partial_{p^-_\alpha}\partial_{p^-_\alpha}\Gamma^{(n)}_{A_\mu G^+ W^-_\mu}(p^+,p^-)\Big|_{p^\pm=0} + 2\;\partial_{p^-_\alpha}\partial_{p^-_\beta}\Gamma^{(n)}_{A_\alpha G^+ W^-_\beta}(p^+,p^-)\Big|_{p^\pm=0} - \\ &\quad - e\,\partial_{p_\alpha}\partial_{p_\alpha}\partial_{p_\mu}\Gamma^{(n)}_{G^+ W^-_\mu}(p)\Big|_{p=0} = 0\,,\end{aligned} \tag{5.6}$$

which is a special case of

$$\begin{aligned}&\left(\partial_{p_\alpha}\partial_{p_\beta}\Gamma^{(n)}_{\hat{V}^a_\gamma \hat{\Phi}_i \hat{V}^b_\nu}\Big|_{p=q=0} + \partial_{p_\beta}\partial_{p_\gamma}\Gamma^{(n)}_{\hat{V}^a_\alpha \hat{\Phi}_i \hat{V}^b_\nu}\Big|_{p=q=0} + \partial_{p_\gamma}\partial_{p_\alpha}\Gamma^{(n)}_{\hat{V}^a_\beta \hat{\Phi}_i \hat{V}^b_\nu}\Big|_{p=q=0}\right) + \\ &+ \sum_j M_{a,j}\;\partial_{p_\alpha}\partial_{p_\beta}\partial_{p_\gamma}\Gamma^{(n)}_{\hat{\Phi}_j \hat{\Phi}_i \hat{V}^b_\nu}(p,q)\Big|_{p=q=0} + f^{abx}\;\partial_{p_\alpha}\partial_{p_\beta}\partial_{p_\gamma}\Gamma^{(n)}_{\hat{\Phi}_i \hat{V}^x_\nu}(-p)\Big|_{p=0} = 0\,.\end{aligned} \tag{5.7}$$

The sum runs over all would-be-Goldstone fields G^0 and $G^\pm$ with masses $M_{\pm,G^\pm} = \pm i M_W$, $M_{Z,G^0} = -M_Z$ and zero for all the other combinations. V^a_α denote the gauge fields, where α runs over the index of the adjoint representation for $\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)$. Here, f^{abx} represent the structure constants of the gauge group in the adjoint representation. Other useful identities can be easily obtained by differentiating with respect to $p_\alpha, p_\beta, q_\gamma$ or $p_\alpha, q_\beta, q_\gamma$ or $q_\alpha, q_\beta, q_\gamma$. Notice that all Green functions appearing in eq. (5.7) are superficially finite.

Having this in mind we can write down the results for $\xi_i^{(n)}, (i = 1,\ldots,6)$

$$\begin{aligned} ie\xi_1^{(n)} &= -ie\xi_6^{(n)} = \frac{1}{18}\left(-6\mathcal{M}_1 - \mathcal{M}_2 + 5\mathcal{M}_3\right), \\ ie\xi_3^{(n)} &= -ie\xi_4^{(n)} = \frac{1}{18}\left(5\,\mathcal{M}_2 - \mathcal{M}_3\right), \\ ie\xi_5^{(n)} &= -ie\xi_2^{(n)} = \frac{1}{18}\left(6\mathcal{M}_1 - \mathcal{M}_2 - \mathcal{M}_3\right), \end{aligned} \tag{5.8}$$

where we have defined

$$
\begin{aligned}
\mathcal{M}_1 &= \frac{iM_W}{16}\,\partial_{p^+_\alpha}\partial_{p^+_\alpha}\Gamma^{(n)}_{A_\mu G^+ W^-_\mu}(p^+,p^-)\Big|_{p^\pm=0}\,,\\
\mathcal{M}_2 &= \frac{iM_W}{16}\,\partial_{p^-_\alpha}\partial_{p^-_\alpha}\Gamma^{(n)}_{A_\mu G^+ W^-_\mu}(p^+,p^-)\Big|_{p^\pm=0}\,,\\
\mathcal{M}_3 &= \frac{iM_W}{8}\,\partial_{p^-_\alpha}\partial_{p^-_\beta}\Gamma^{(n)}_{A_\alpha G^+ W^-_\beta}(p^+,p^-)\Big|_{p^\pm=0}\,.
\end{aligned}
\tag{5.9}
$$

Note that it is also possible to find other representations of the results. However, they are equivalent after exploiting eq. (5.7).

Finally, the symmetrical Green function reads

$$
\begin{aligned}
\mathrm{I\!\Gamma}^{(n)}_{\hat{A}_\rho \hat{W}^+_\mu \hat{W}^-_\nu}(p^+,p^-) = \Gamma^{(n)}_{\hat{A}_\rho \hat{W}^+_\mu \hat{W}^-_\nu}(p^+,p^-) - \Big[T^2_{p^+,p^-}\Gamma^{(n)}_{\hat{A}_\rho \hat{W}^+_\mu \hat{W}^-_\nu}(p^+,p^-) \,+ \\
+\, \Xi^{(n)}_{\hat{A}_\rho \hat{W}^+_\mu \hat{W}^-_\nu}(p^+,p^-)\Big]\,.
\end{aligned}
\tag{5.10}
$$

Let us summarize the steps which that to be performed in order to compute the $\mathcal{O}(\alpha\alpha_s)$ corrections to the AWW vertex. The basic equation is eq. (1.5). In a first step the two-loop amplitude, denoted by $\Gamma^{(2)}_{AWW}$, has to be calculated using a specific regularization scheme and a specific subtraction scheme. There are two contributions: genuine two-loop diagrams and finite one-loop counterterm diagrams. Clearly, the same regularization scheme has to be used in the two contributions. The divergent parts in both contributions are assumed to be subtracted already. The finite one-loop counterterm contributions are two-fold: first, the one-loop counterterm due to the Taylor-subtraction and second, the universal one-loop counterterms (see eqs. (2.5)–(2.9)).

5.2 The case ZWW

Having the physical process $e^+e^- \to WW$ in mind, we also have to discuss the QCD corrections to the ZWW vertex within the channel $e^+e^- \to Z \to WW$. The case ZWW has some similarity to AWW. However, due to the connection of the Z boson and the neutral Goldstone boson with the finite Z boson mass, both the identities and the analysis, get more involved.

The equations corresponding to (5.1) and (5.2) read:

$$
\begin{aligned}
& i\,(p_+ + p_-)^\mu\,\Gamma^{(n)}_{\hat{Z}_\mu \hat{W}^+_\rho \hat{W}^-_\sigma}(p_+,p_-) - M_Z\Gamma^{(n)}_{\hat{G}^0 \hat{W}^+_\rho \hat{W}^-_\sigma}(p_+,p_-) \,+ \\
&\qquad + ie\frac{c_W}{s_W}\left(\mathrm{I\!\Gamma}^{(n)}_{\hat{W}^+_\rho \hat{W}^-_\sigma}(p_-) - \mathrm{I\!\Gamma}^{(n)}_{\hat{W}^+_\rho \hat{W}^-_\sigma}(-p_+)\right) = \Delta'^{,(n)}_{W,\lambda_Z \hat{W}^+_\rho \hat{W}^-_\sigma}(p_+,p_-)\,,\\
& i\,(p_+ + p_-)^\mu\,\Gamma^{(n)}_{\hat{Z}_\mu \hat{G}^+ \hat{G}^-}(p_+,p_-) - M_Z\Gamma^{(n)}_{\hat{G}^0 \hat{G}^+ \hat{G}^-}(p_+,p_-) \,+ \\
&\qquad + ie\frac{c_W^2 - s_W^2}{2\,s_W c_W}\left(\mathrm{I\!\Gamma}^{(n)}_{\hat{G}^+ \hat{G}^-}(p_-) - \mathrm{I\!\Gamma}^{(n)}_{\hat{G}^+ \hat{G}^-}(-p_+)\right) = \Delta'^{,(n)}_{W,\lambda_Z \hat{G}^+ \hat{G}^-}(p_+,p_-)\,,
\end{aligned}
$$

$$
\begin{aligned}
&i\,(p_+ + p_-)^\mu\, \Gamma^{(n)}_{\hat{Z}_\mu \hat{W}^+_\rho \hat{G}^-}(p_+,p_-) - M_Z \Gamma^{(n)}_{\hat{G}^0 \hat{W}^+_\rho \hat{G}^-}(p_+,p_-) \;+ \\
&+ ie \left(\frac{c_W}{s_W} \mathrm{I}\!\Gamma^{(n)}_{\hat{W}^+_\rho \hat{G}^-}(p_-) - \frac{c_W^2 - s_W^2}{2\, s_W c_W} \mathrm{I}\!\Gamma^{(n)}_{\hat{W}^+_\rho \hat{G}^-}(-p_+) \right) = \Delta'^{,(n)}_{W,\lambda_A \hat{W}^+_\rho \hat{G}^-}(p_+,p_-)\,,
\end{aligned}
\tag{5.11}
$$

and

$$
\begin{aligned}
&-ip^\rho_+ \Gamma^{(n)}_{\hat{Z}_\mu \hat{W}^+_\rho \hat{W}^-_\sigma}(p_+,p_-) + iM_W \Gamma^{(n)}_{\hat{Z}_\mu \hat{G}^+ \hat{W}^-_\sigma}(p_+,p_-) \;- \\
&- ie \left(\frac{c_W}{s_W} \mathrm{I}\!\Gamma^{(n)}_{\hat{W}^+_\mu \hat{W}^-_\sigma}(p_-) + \mathrm{I}\!\Gamma^{(n)}_{\hat{Z}_\mu \hat{A}_\sigma}(-p_+ - p_-) - \frac{c_W}{s_W} \mathrm{I}\!\Gamma^{(n)}_{\hat{Z}_\mu \hat{Z}_\sigma}(-p_+ - p_-) \right) = \\
&\quad = \Delta'^{,(n)}_{W,Z_\mu \lambda_+ \hat{W}^-_\sigma}(p_+,p_-)\,, \\
&-ip^\rho_+ \Gamma^{(n)}_{\hat{Z}_\mu \hat{W}^+_\rho \hat{G}^-}(p_+,p_-) + iM_W \Gamma^{(n)}_{\hat{Z}_\mu \hat{G}^+ \hat{G}^-}(p_+,p_-) \\
&+ e \left(-i \frac{c_W}{s_W} \mathrm{I}\!\Gamma^{(n)}_{\hat{W}^+_\mu \hat{G}^-}(p_-) + \frac{1}{2 s_W} \mathrm{I}\!\Gamma^{(n)}_{\hat{Z}_\mu \hat{G}^0}(-p_+ - p_-) - \frac{i}{2 s_W} \mathrm{I}\!\Gamma^{(n)}_{\hat{Z}_\mu \hat{H}}(-p_+ - p_-) \right) = \\
&\quad = \Delta'^{,(n)}_{W,Z_\mu \lambda_+ \hat{G}^-}(p_+,p_-)\,.
\end{aligned}
\tag{5.12}
$$

Again all equations that can be obtained by interchanging W^+ and W^- are omitted.

In order to keep the discussion simpler we restrict ourselves to the one- and two-loop level. This means in the following equations the index n is either 1 or 2. The second-order Taylor subtraction leads to

$$
\begin{aligned}
\Psi'^{,(n),W}_{\lambda_Z \hat{W}^+_\rho \hat{W}^-_\sigma}(p_+,p_-) &= -\frac{M_Z}{2} \left(p^\mu_+ p^\nu_+ \partial_{p^\mu_+} \partial_{p^\nu_+} + 2\, p^\mu_+ p^\nu_- \partial_{p^\mu_+} \partial_{p^\nu_-} + p^\mu_- p^\nu_- \partial_{p^\mu_-} \partial_{p^\nu_-} \right) \times \\
&\quad \times \Gamma^{(n)}_{\hat{G}^0 \hat{W}^+_\rho \hat{W}^-_\sigma}(p_+,p_-) \Big|_{p_\pm = 0} - \\
&\quad - \frac{ie\, c_W}{s_W} \Big[\left(\xi^{(n)}_{\hat{W},1}\, p^2_- g_{\rho\sigma} + \xi^{(2)}_{\hat{W},2}\, p_{-,\rho} p_{-,\sigma} \right) - \Big(\xi^{(n)}_{\hat{W},1}\, p^2_+ g_{\rho\sigma} + \\
&\qquad\qquad + \xi^{(n)}_{\hat{W},2}\, p_{+,\rho} p_{+,\sigma} \Big) \Big]\,, \\
\Psi'^{,(n),W}_{\lambda_Z \hat{G}^+ \hat{G}^-}(p_+,p_-) &= 0\,, \\
\Psi'^{,(n),W}_{\lambda_Z \hat{W}^+_\rho \hat{G}^-}(p_+,p_-) &= 0\,, \\
\Psi'^{,(n)}_{\hat{Z}_\mu \lambda_+ \hat{W}^-_\sigma}(p_+,p_-) &= \frac{i\, M_W}{2} \left(p^\rho_+ p^\nu_+ \partial_{p^\rho_+} \partial_{p^\nu_+} + 2\, p^\rho_+ p^\nu_- \partial_{p^\rho_+} \partial_{p^\nu_-} + p^\rho_- p^\nu_- \partial_{p^\rho_-} \partial_{p^\nu_-} \right) \times \\
&\quad \times \Gamma^{(n)}_{\hat{Z}_\mu \hat{G}^+ \hat{W}^-_\sigma}(p_+,p_-) \Big|_{p_\pm = 0} + ie \frac{c_W}{s_W} \left(\xi^{(n)}_{\hat{W},1} p^2_- g_{\mu\sigma} + \xi^{(n)}_{\hat{W},2} p_{-,\mu} p_{-,\sigma} \right) - \\
&\quad - ie \frac{c_W}{s_W} \left(\xi^{(n)}_{\hat{Z},1} (p_- + p_+)^2\, g_{\mu\sigma} + \xi^{(n)}_{\hat{Z},2} (p_- + p_+)_\mu (p_- + p_+)_\sigma \right), \\
\Psi'^{,(n)}_{Z_\mu \lambda_+ \hat{G}^-}(p_+,p_-) &= 0\,.
\end{aligned}
\tag{5.13}
$$

In general the breaking term $\Psi^{(n),W}_{\lambda_Z \hat{W}^+_\rho \hat{G}^-}$ does not vanish. However, as we only consider two-loop QCD corrections it is zero. In this approximation also the Green function $\Gamma^{(n)}_{\hat{G}^0 \hat{G}^+ \hat{G}^-}(p_+,p_-)$ (and thus $\Psi'^{,(n),W}_{\lambda_Z \hat{G}^+ \hat{G}^-}(p_+,p_-)$) vanishes as we checked by an explicit calculation. This is essentially due to the invariance under CP transformations of the bosonic sector. Note that starting form three loops, the CP violation induced by the CKM mixings will generate some CP violating bosonic counterterms.

The most general counterterm that can be used to re-absorb the breaking term of the WTIs reads

$$\Xi^{(n)}_{\hat{Z}_\mu \hat{W}^+_\rho \hat{W}^-_\sigma}(p^+,p^-) = e\Big[g_{\mu\rho}\Big(\xi^{(n)}_7 p^+_\sigma + \xi^{(n)}_8 p^-_\sigma\Big) + g_{\rho\sigma}\Big(\xi^{(n)}_9 p^+_\mu + \xi^{(n)}_{10} p^-_\mu\Big) + \\ + g_{\mu\sigma}\Big(\xi^{(n)}_{11} p^+_\rho + \xi^{(n)}_{12} p^-_\rho\Big)\Big]\,. \tag{5.14}$$

Inserting $\Xi^{(n)}_{\hat{Z}_\mu \hat{W}^+_\nu \hat{W}^-_\rho}(p^+,p^-)$ in the above identities and comparing the coefficients with the breaking terms leads to equations from which the coefficients $\xi^{(n)}_i (i = 7,\ldots,12)$ can be determined. One possible set of equations reads

$$\begin{aligned}
ie\Big(4\xi^{(n)}_9 + \xi^{(n)}_7 + \xi^{(n)}_{11}\Big) &= -\frac{1}{8}\partial_{p^+_\alpha}\partial_{p^+_\alpha}\mathcal{M}_{\beta\beta}\,,\\
ie\Big(5\xi^{(n)}_7 + 5\xi^{(n)}_{11} + 2\xi^{(n)}_9\Big) &= -\frac{1}{4}\partial_{p^+_\alpha}\partial_{p^+_\beta}\mathcal{M}_{\alpha\beta}\,,\\
ie\Big(5\xi^{(n)}_7 + 5\xi^{(n)}_9 + 2\xi^{(n)}_{11}\Big) &= \frac{1}{4}\partial_{p^+_\alpha}\partial_{p^+_\beta}\mathcal{N}_{\alpha\beta}\,,\\
ie\xi_8 &= \frac{1}{72}\Big(5\partial_{p^-_\alpha}\partial_{p^+_\beta} - \partial_{p^+_\alpha}\partial_{p^-_\beta} - \partial_{p^+_\gamma}\partial_{p^-_\gamma}g^{\alpha\beta}\Big)\mathcal{M}_{\alpha\beta}\,,\\
ie\xi_{10} &= \frac{1}{72}\Big(5\partial_{p^+_\alpha}\partial_{p^-_\beta} - \partial_{p^-_\alpha}\partial_{p^+_\beta} - \partial_{p^-_\gamma}\partial_{p^+_\gamma}g^{\alpha\beta}\Big)\mathcal{M}_{\alpha\beta}\,,\\
ie\xi_{12} &= \frac{1}{72}\Big(5\partial_{p^+_\gamma}\partial_{p^-_\gamma}g^{\alpha\beta} - \partial_{p^-_\alpha}\partial_{p^+_\beta} - \partial_{p^+_\alpha}\partial_{p^-_\beta}\Big)\mathcal{M}_{\alpha\beta}\,,
\end{aligned} \tag{5.15}$$

where we introduced the short-hand notation $\mathcal{M}_{\alpha\beta} = \Psi^{(n),W}_{\lambda_Z \hat{W}^+_\alpha \hat{W}^-_\beta}(p_+,p_-)$ and $\mathcal{N}_{\alpha\beta} = \Psi^{(n)}_{\hat{Z}_\alpha \lambda_+ \hat{W}^-_\beta}(p_+,p_-)$. It is understood, that after the differentiation the momenta are set to zero. Note that eqs. (5.15) are only unique up the consistency conditions (similar to the one in eq. (5.7)) among the WTIs.

As in the previous case, the wave-function renormalization of the background field $\hat{Z}$ is fixed by the WTIs. In particular, in our analysis the θ_W-angle is fixed by the on-shell condition $M_W/M_Z = c_W$, where M_W and M_Z are the physical masses. Actually, using the above equations one obtains two equations which fix $\xi^{(n)}_{\hat{Z},i} (i = 1,2)$

$$\begin{aligned}
\xi_{\hat{Z},1} &= \xi_{\hat{W},1} + \frac{s_W M_W}{36 e c_W}\Big(-\frac{1}{4}\partial_{p^-_\mu}\partial_{p^-_\sigma} + \frac{5}{8}\partial_{p^-_\alpha}\partial_{p^-_\alpha}g^{\mu\sigma}\Big)\Gamma_{\hat{Z}_\mu \hat{G}^+ \hat{W}^-_\sigma}(p_+,p_-)\Big|_{p_\pm=0}\,,\\
\xi_{\hat{Z},2} &= \xi_{\hat{W},2} + \frac{s_W M_W}{36 e c_W}\Big(\partial_{p^-_\mu}\partial_{p^-_\sigma} - \frac{1}{4}\partial_{p^-_\alpha}\partial_{p^-_\alpha}g^{\mu\sigma}\Big)\Gamma_{\hat{Z}_\mu \hat{G}^+ \hat{W}^-_\sigma}(p_+,p_-)\Big|_{p_\pm=0}\,,
\end{aligned} \tag{5.16}$$

where $\xi^{(n)}_{\hat{W},i}$ is given in eq. (5.5). Again, using the consistency conditions, one can check that other possible equations are not independent.

Finally we want to remark that at higher orders also other counterterms may be involved in the analysis. However, this depends on the specific type of radiative corrections which are taken into account. In particular, $\Gamma^{(3)}_{\hat{G}^0\hat{G}^+\hat{G}^-}$ is only needed if three-loop electro-weak corrections are considered.

6. IR re-shuffling

In practical applications quark masses can often be neglected. However, through the Taylor subtraction this can in general induce IR divergences in the corresponding two- and three-point functions. On the other hand it is important that the breaking terms are IR-finite. In this section we discuss the modifications of our procedure needed to deal with these cases. In a first step we want to approach the problem from a more theoretical point of view and only then apply it to the case of AWW.

A careful analysis of the off-shell IR problems in the SM [14, 13] shows that suitable normalization conditions are sufficient to guarantee the IR finiteness of Green functions in case non-exceptional momenta are chosen. However, Taylor subtraction around zero external momenta may cause problems. Let us assume that only the highest order of the derivative leads to IR divergences. This means that for a given Green function Γ, with UV divergence degree ω, $T^\omega\Gamma$ is IR-divergent but $T^{\omega-1}\Gamma$ is not. It is then tempting not to perform the complete Taylor expansion and leave out the term with highest power of derivative. This modifies the subtraction scheme presented in [20, 6] as discussed in the following.

Acting on a broken WTI such as eq. (1.2) with the Taylor operator $(1 - T^\delta)$, we obtain

$$(1 - T^\delta)\mathcal{W}_{(\lambda)}\left(\Gamma^{(n)}\right) = 0\,. \tag{6.1}$$

After commuting the Taylor operator $(1 - T^\delta)$ with $\mathcal{W}_{(\lambda)}$ this transforms to

$$\mathcal{W}_{(\lambda)}\left[(1 - T^{\delta'})\Gamma^{(n)}\right] = \left[T^\delta\mathcal{W}_{(\lambda)} - \mathcal{W}_{(\lambda)}T^{\delta'}\right] \equiv \hbar^n\Psi^{(n)}_W(\lambda)\,, \tag{6.2}$$

where δ' is the naive power counting degree of $\Gamma^{(n)}$. In general, one has $\delta \geq \delta'$, Therefore the commutation of the Taylor operator with $\mathcal{W}_{(\lambda)}$ leads to over-subtractions of $\Gamma^{(n)}$ and, thus, to the new breaking terms $\Psi^{(n)}_W(\lambda)$ occurring on the r.h.s. of eq. (6.2) (for more details see [6]).

In eq. (6.2) both terms, $T^{\delta'}\Gamma^{(n)}$ and $\Psi^{(n)}_W(\lambda)$, could be IR divergent. This suggests a re-shuffling of the terms from the r.h.s. to the l.h.s., which defines a new breaking term through

$$\begin{aligned}\mathcal{W}_{(\lambda)}\left[(1 - T^{\delta_{\rm IR}})\Gamma^{(n)}\right] &= \left[T^\delta\mathcal{W}_{(\lambda)} - \mathcal{W}_{(\lambda)}T^{\delta'}\right] + \mathcal{W}_{(\lambda)}\left[(T^{\delta'} - T^{\delta_{\rm IR}})\Gamma^{(n)}\right]\\ &\equiv \hbar^n\Psi'^{(n)}_W(\lambda)\,.\end{aligned} \tag{6.3}$$

Note that $\delta_{\rm IR} = \delta' - 1$ is used for the IR-divergent Green functions. In this way all terms of eq. (6.3) are IR-safe. The price of the re-shuffling is that the expression for $\Psi'^{(n)}_W(\lambda)$ becomes in general more complicated than the original breaking term, $\Psi^{(n)}_W(\lambda)$. However, its computation is still simpler than $\Delta^{(n)}_W(\lambda)$ of eq. (1.2). The only requirement is an intermediate IR regulator, needed for the evaluation of the individual Green functions appearing in $\Psi'^{(n)}_W(\lambda)$.

We also have to mention that the new breaking terms $\Psi'^{(n)}_W(\lambda)$ and the corresponding counterterms $\Xi^{(n)}_W$ depend on the UV subtraction. In fact, since $\delta_{\rm IR} = \delta' - 1$, some Green functions are only superficially finite because of the UV subtraction. This implies that $\Psi'^{(n)}_W(\lambda)$ as well as the final counterterm $-T^{\delta_{\rm IR}}\Gamma^{(n)} + \Xi'^{(n)}$ (see eq. (1.7)) depends on the computation details. In addition, the dependence on the UV regulator of the breaking terms and the corresponding non-invariant counterterms breaks the universality of the computation (see [6]).

As an explicit example let us consider the case AWW. In particular, we specify to the top-bottom doublet and neglect the mass of the bottom quark. In this case the one-loop sub-divergences (cf. section. 2) become IR-divergent. Using the IR re-shuffling discussed above eq. (2.2) transforms to

$$
\begin{aligned}
\Psi'^{(1)}_{\lambda_A \bar{b} b}(p,q) &= i\,(p+q)^\mu\, T^0_{p,q} \Gamma^{(1)}_{\hat{A}_\mu \bar{b} b}(p,q) + ieQ_q \Big((T^1_q - T^0_q)\Gamma^{(1)}_{\bar{b}b}(q) - \\
&\qquad - (T^1_p - T^0_p)\Gamma^{(1)}_{\bar{b}b}(-p) \Big)\,, \qquad (6.4)\\
\Psi'^{(1)}_{\lambda_+ \bar{t} b}(p_t,p_b) &= i\,M_W \left(T^1_{p_t p_b} - T^0_{p_t p_b}\right) \Gamma^{(1)}_{\hat{G}^+ \bar{t} b}(p_t,p_b) - \frac{ie}{s_W\sqrt{2}} P_R V_{tb} (T^1_{p_b} - T^0_{p_b})\Gamma^{(1)}_{\bar{b}b}(p_b)\,.
\end{aligned}
$$

Of course, no IR divergences appear for the $A\bar{t}t$ vertex and thus we still have $\Psi^{(1)}_{\lambda_A \bar{t} t} = 0$. Notice that the advantages due to the zero-momentum subtractions have only slightly been reduced. The computation of the breaking terms still relies on Green functions expanded around zero external momenta. We also stress that the proposed rearrangement solves the spurious IR problem due to Taylor subtractions in general.

The problem of IR divergences appears also at two loops. In that case, one has to handle the intermediate regularization procedure with some care. For completeness concerning the AWW description, we present an example of IR re-shuffling at two loops.

It is easy to see that the zero-momentum subtraction of the first of eqs. (5.2) and the breaking term $\Psi^{(2)}_{A_\mu \lambda_+ \hat{W}^-_\sigma}(p_+, p_-)$ are IR-divergent in the approximation that $m_b = 0$. So, we have to recombine the Green functions in such a way that the computation is IR-safe. For that purpose, we can notice that also the zero-momentum subtraction of $\mathrm{I\!\Gamma}^{(2)}_{\hat{A}_\mu \hat{Z}_\sigma}$ and $\mathrm{I\!\Gamma}^{(2)}_{\hat{A}_\mu \hat{A}_\sigma}$ produces IR divergence. Therefore, the most natural

IR-safe recombination is

$$
\begin{aligned}
\Psi'^{,(2)}_{A_\mu \lambda_+ \hat{W}^-_\sigma}(p_+, p_-) = \frac{i\, M_W}{2} & \Big(p^\rho_+ p^\nu_+ \partial_{p^\rho_+} \partial_{p^\mu_+} + 2\, p^\rho_+ p^\nu_- \partial_{p^\rho_+} \partial_{p^\nu_-} + \\
& + p^\rho_- p^\nu_- \partial_{p^\rho_-} \partial_{p^\nu_-} \Big)\, \Gamma^{(2)}_{\hat{A}_\mu \hat{G}^+ \hat{W}^-_\sigma}(p_+, p_-) \Big|_{p_\pm = 0} - \\
& - ie \left(\xi^{(n)}_{\hat{W},1}\, p^2_- g_{\rho\sigma} + \xi^{(n)}_{\hat{W},2}\, p_{-,\rho} p_{-,\sigma} \right) - \\
& - ie \left(T^2_q \mathrm{I\Gamma}^{(2)}_{\hat{A}_\mu \hat{A}_\sigma}(-q) + -\frac{c_W}{s_W} T^2_q \mathrm{I\Gamma}^{(2)}_{\hat{A}_\mu \hat{Z}_\sigma}(-q) - \right), \qquad (6.5)
\end{aligned}
$$

where $q = p_+ + p_-$. $\Psi'^{,(2)}_{A_\mu \lambda_+ \hat{W}^-_\sigma}$ is IR-safe and, thus, can be used to compute the counterterms.

7. Conclusion

The techniques developed in [6] have been applied to the three-gauge-boson vertices. In the framework of the BFM all functional identities are derived at the n-loop order. Since in the SM there exists no invariant regularization scheme (besides the lattice regularization with the Ginsparg-Wilson version of chiral symmetry) the functional identities are broken by local terms. Most of them are simply removed by the application of the Taylor operator [6]. The analysis of the remaining ones is presented up to the two-loop level, where additional QCD corrections to the one-loop fermionic diagrams are considered. Finally subtleties in connection to IR divergences resulting for the expansion around zero external momenta are discussed in detail.

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